SHOCK WAVES IN CONICAL FLOWS

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In the papers [1 to 3], the author showed the impossibility of a continuous flow near the "top" of a triangular plate, or other conical bodies, in a uniform supersonic flow. These conclusions were based on the properties of simple conical waves and on arguments connected with the formulation of boundary value problems for mixed equations of the elliptic-hyperbolic type. Below will be given another approach to the problem, based on the asymptotic representations of the solution in "boundary layers" near weak shock waves and Mach cones for uniform flow [4]. In the example of the flow near the "top" of the triangular plate, it is established that the appearance (or absence) of shock waves is determined by the behavior of the usual linear solutions near the corresponding boundary. Asymptotic formulas are found, defining the position of the shock wave for the "top" of the triangular plate.

1. We consider a triangular plate at an angle of attack δ at rest in an inviscid gas flow with velocity W_1 and Mach nymber M_1. We assume that the edges of the wing are supersonic, so that the conical flows resulting from flow past "above" and "below" the wing do not interact and may be considered separately (Fig.1).

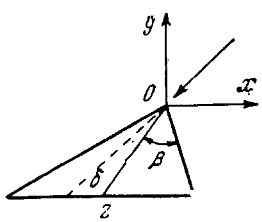

Fig.1

In conical flows, the velocity components u, v, w, along the Cartesian coordinates, the entropy S,,and the pressure p all depend only on angular coordinates, for which we take $\xi = x/z$ and $\eta = y/z$, the z-axis being directed along the axis of symmetry of the wing (Fig.1). For irrotational conical flows, the conical potential $F(\xi, \eta) = z^{-1}\varphi(x, y, z)$ (where φ is the velocity potential) satisfies Equation

$$L[F] = AF_{\xi\xi} + 2BF_{\xi\eta} + CF_{\eta\eta} = 0 \tag{1.1}$$

$$A = a^2(1 + \xi^2) - (u - \xi w)^2$$

$$B = a^2\xi\eta - (u - \xi w)(v - \eta w)$$

$$C = a^2(1 + \eta^2) - (v - \eta w)^2$$

$$a^2 = a_1^2 - {}^1/_2(\gamma - 1)(u^2 + v^2 + w^2 - W_1^2)$$

Here a_1 is the speed of sound in the unperturbed flow, a is the local speed of sound, and γ the adiabatic exponent.

Moreover,

$$u = F_\xi, \qquad v = F_\eta, \qquad w = F - \xi F_\xi - \eta F_\eta \tag{1.2}$$

In what follows, polar coordinates will be needed in the plane; intoducing them by Formulas $\xi = r\cos\theta$, $\eta = r\sin\theta$, we then have

$$u = \cos\theta F_r - \frac{\sin\theta}{r} F_\theta, \qquad v = \sin\theta F_r + \frac{\cos\theta}{r} F_\theta, \qquad w = F - rF_r \tag{1.3}$$

By virtue of the symmetry, e show the flow pattern for the "top" of the plate at small angles of attack δ in the $\xi\eta$-plane for the case when $\xi > 0$ (Fig.2). The wing is represented by the segment 0-3 . The envelope of the Mach cones of unperturbed flow with vertices on the lateral edge is given by the arc 1-2 (an arc of the Mach cone with vertex at the point 0 in Fig.1) and the straight line 2-3. At the flow past the lateral edge of the plate the Prandtl-Meyer flow is formed, followed by a uniform flow adjacent to the wing surface, region 3-4-5-6-3 . The Prandtl-Meyer flow has a family of straight characteristics of Equation (1.1) passing through the point 3 (straight lines 3-2 and 3-6). The region of general conical flow above the center portion of the wing (if the flow is assumed continuous), is bounded by the arc of the Mach cone 1-2 , the curved characteristic of the Prandtl-Meyer flow 2-6 , the straight characteristic 6-5 , and the arc of the Mach cone 5-4 for the uniform flow in region 3-4-5-6-3 . We shall not decide now the existence of the shocks 2-7 and 2-8 . From what follows, it will become clear that 2-8 degenerates into the arc of the Mach cone 1-2 , while 2-7 exists. An asymptotic formula for determining its position (for $\delta \to 0$) will be given .

2. In [4] the author pointed out that near the weak shock waves and the Mach cones of uniform flow there exist "boundary leayers", in which the solution of (1.1) cannot be represented by a usual power series in δ , and the corresponding expansions were given. Our plan of procedure is as follows. Inside the region 0-1-2-6-5-4-0 (Fig.2) the solution may be represented as a power series in δ .

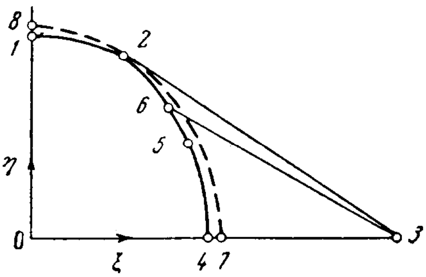

Fig. 2

We take the first term of this expansion (linear theory) and attempt to "match" it to the first term of the solution resulting from the "boundary layer" near 1-2 (or 2-8) and 2-7 (or 4-5) .

In this manner, it will be established that the different expansions may be matched only if it is assumed that the boundary of the general conical flow region is 1-2 and 2-7, i.e. the shock 2-8 is absent, while the shock 2-7 occurs. (We note that as $\delta \to 0$, the arc 2-6-5 vanishes and the entire flow is determined by the arcs 1-2 and 5-4.)

The coordinate system of Fig.1 will be called the basic system. We introduce two more auxiliary coordinate systems. In the first system, the z^*-axis is taken along the unperturbed flow velocity, and all quantities in this coordinate system will be denoted by a star superscript. In the second system, the z°-axis is taken along the uniform flow velocity in the region 3-4-5-6-3 and all quantities are denoted by the superscript $^\circ$. We now write the expansion for the conical potential F resulting from the "boundary layer" [4] near 1-2 , setting $\lambda^{1/2} = \delta,\ \psi(\theta^*, \lambda) = \psi_0(\theta^*) + \ldots,$

$$F^*(r^*, \theta^*) = W_1 + \delta\left[\frac{8\sqrt{3}}{9}\psi_0^{1/2}(\theta^*)\frac{W_1}{\gamma+1}\frac{m_1^4}{M_1^4}(r_1^* - r^*)^{3/2} + \ldots\right] + O(\delta^2)$$

$$(m_1 = (M_1^2 - 1)^{1/2}, \quad r_1^* = m_1^{-1}) \tag{2.1}$$

If we assume that near 1-2 a shock 2-8 exists, then we must substitute $\varphi_0^{1/2}(\theta^*)$ for $-\psi_0^{1/2}(\theta^*)$ in (2.1). (The position of 2-8 is determined by Formula $r^* = r_1^* + \delta^2\psi_0(\theta^*) + \ldots$) In (2.1) we change to the basic coordinates. The first auxiliary coordinate system is obtained from the basic system by means of a rotation about the y-axis through an angle δ . Writing the transformation formula from x^*, y^*, z^* to x, y, z, we easily get

$$r^* = r - \delta \sin\theta\,(1 + r^2) + O(\delta^2), \qquad \theta^* = \theta - \delta r^{-1}\cos\theta + O(\delta^2) \qquad (2.2)$$

Moreover, the conical potential F in the basic coordinate system is obtained from F^* by Formula

$$F(r, \theta) = F^*(r^*, \theta^*)\sqrt{(1 + r^2)/(1 + r^{*2})}$$

Substituting here (2.1),(2.2) and expanding the result in δ, we get

$$F = W_1 + \delta\Big[r_1 W_1 \sin\theta - W_1 \sin\theta\,(r_1 - r) + \frac{8\sqrt{3}}{9}\frac{W_1}{\gamma + 1}\frac{m_1^4}{M_1^4}\psi_0^{1/2}(\theta)\,(r_1 - r)^{3/2} + \ldots\Big] + O(\delta^2) \qquad (2.3)$$

where $r_1 = r_1^*$. According to (1.3),

$$w = W_1\Big\{1 + \delta\Big[\frac{4\sqrt{3}}{3}\frac{m_1^3}{(\gamma + 1)M_1^4}\psi_0^{1/2}(\theta)\,(r_1 - r)^{1/2} + \ldots\Big] + O(\delta^2)\Big\} \qquad (2.4)$$

We carry out the same operations for the flow near the shock 2-7 (or arc 5-4). We write the expansion for F resulting from the "boundary layer" near 2-7 [4], setting $\lambda^{1/2} = \delta$, $\varphi(\theta^\circ, \lambda) = \varphi_0(\theta^\circ) + \ldots$,

$$F^\circ(r^\circ, \theta^\circ) = W_1^\circ + \delta\Big[-\frac{\varphi_0^{1/2}(\theta^\circ)}{\gamma + 1}W_1^\circ\frac{m_1^{\circ 4}}{M_1^{\circ 4}}\frac{8\sqrt{3}}{9}(r_1^\circ - r^\circ)^{3/2} + \ldots\Big] + O(\delta^2) \qquad (2.5)$$

$$(m_1^\circ = \sqrt{M_1^{\circ 2} - 1}, \quad r_1^\circ = 1/m_1^\circ)$$

Here W_1° and M_1° are the velocity and Mach number in the region 3-4-5-6-3 (Fig.2).

The second auxiliary coordinate system is obtained from the basic coordinate system by a rotation about the y-axis through an angle σ, which is a function of δ, β, M_1, and γ. We represent it as a series in the powers of δ

$$\sigma = \sigma_1\delta + \sigma_2\delta^2 + \ldots$$

We do exactly the same for the other quantities

$$W_1^\circ = W_1 + W_{11}\delta + \ldots, \quad M_1^{\circ 2} = M_1^2 + O(\delta), \quad m_1^{\circ 2} = m_1^2 + O(\delta), \quad r_1^\circ = r_1 + O(\delta)$$

etc. Moreover,

$$F(r, \theta) = F^\circ(r^\circ, \theta^\circ)\sqrt{\frac{1 + r^2}{1 + r^{\circ 2}}}$$

and r°, θ° are related to r, θ by Formulas

$$r^\circ = r + \delta\sigma_1\cos\theta\,(1 + r^2) + O(\delta^2), \qquad \theta^\circ = \theta - \delta\sigma_1\frac{\sin\theta}{r} + O(\delta^2)$$

Substituting (2.5), r°, θ° into $F(r, \theta)$, expanding the result in δ, we get

$$F = W_1 + \delta\Big[W_{11} - W_1\sigma_1 r_1\cos\theta + W_1\sigma_1\cos\theta\,(r_1 - r) - \frac{8\sqrt{3}}{9}\frac{W_1}{\gamma + 1}\frac{m_1^4}{M_1^4}\varphi_0^{1/2}(\theta)\,(r_1 - r)^{3/2} + \ldots\Big] + O(\delta^2)$$

From this, using (1.3), we find

$$w = W_1\Big\{1 + \delta\Big[\frac{W_{11}}{W_1} - \frac{4\sqrt{3}}{3}\frac{m_1^3}{(\gamma + 1)M_1^4}\varphi_0^{1/2}(\theta)\,(r_1 - r)^{1/2} + \ldots\Big] + O(\delta^2)\Big\} \qquad (2.6)$$

If we assume that the shock 2-7 is absent, then in Formula (2.6) the quantity $\varphi_0^{1/2}(\theta)$ must be replaced by $-\psi_0^{1/2}(\theta)$.

In the inner part of the region 0-1-2-7-0, the quantity w is expressed as

$$w = W_1\,[1 + \delta w_1(\alpha, \theta) + O(\delta^2)] \qquad (2.7)$$

where $w_1(\alpha, \theta)$ is a harmonic function of the polar coordinates α, θ and

$r/r_1 = 2\alpha/(1+\alpha^2)$ ($r = r_1$ corresponds to $\alpha = 1$). Transforming (2.4) and (2.6) to the variables α, θ, we get

$$w = W_1\left\{1 + \delta\left[\frac{2\sqrt{6}}{3}\,\frac{m_1^{5/2}}{(\gamma+1)M_1^4}\psi_0^{1/2}(\theta)(1-\alpha) + \ldots\right] + O(\delta^2)\right\} \qquad (2.8)$$

$$w = W_1\left\{1 + \delta\left[\frac{W_{11}}{W_1} - \frac{2\sqrt{6}}{3}\,\frac{m_1^{5/2}}{(\gamma+1)M_1^4}\varphi_0^{1/2}(\theta)(1-\alpha) + \ldots\right] + O(\delta^2)\right\} \qquad (2.9)$$

3. We shall argue in the following fashion.

The expansion (2.7) represents an exact solution in the interior of the circle $\alpha \leqslant 1$. In the neighborhood of $\alpha = 1$, this series diverges, but each term of this series is defined at $\alpha \leqslant 1$, so that as $\alpha \to 1$ this function must have a completely determined behavior. This behavior of $w_1(\alpha,\theta)$ is given by the coefficients of δ in the Formulas (2.8) and (2.9). Letting α tend to unity, we obtain from (2.8) and (2.9) that on the arc of the unit circle 1-2 of Fig.3 (corresponding to 1-2 in Fig.2 or to the shock 8-2), $w_1 = 0$, while on the arc 2-7 of Fig.3 (corresponding to the shock 2-7 in Fig.2 or to the Mach cone 4-5), $w_1 = W_{11}/W_1$; i.e. in the $\alpha\theta$-plane, the usual linear problem is obtained exactly.

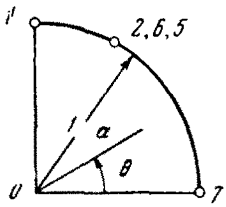

Fig. 3

After finding $w_1(\alpha,\theta)$ we "match" the derivative w_α found from the linear theory with the w_α given by the expansions (2.8) and (2.9) for $\alpha = 1$. Since according to the linear theory $w_\alpha < 0$ on arc 1-2 of Fig.3 and $w_\alpha > 0$ on arc 2-7 of Fig.3, then "matching" is possible only if we assume that the boundary of the conical flow region is composed of the arc of the Mach cone 1-2 of Fig.2 and the shock 2-7 (Fig.2). From the matching condition, we find $\psi_0^{1/2}(\theta)$ and $\varphi_0^{1/2}(\theta)$, which define the velocity of the flow expansion in the neighborhood of 1-2 as well as determine the position of the shock 2-7. In particualr, for $\varphi_0(\theta)$, we have

$$\varphi_0(\theta) = \frac{3}{8}\,\frac{(\gamma+1)^2 M_1^8}{m_1^5}\,[(w_{1\alpha})_{\alpha=1}]^2 \qquad (3.1)$$

and the position of the shock 2-7, Fig.2, is given in the second auxiliary system by

$$r^\circ = r_1^\circ + \delta^2\varphi_0(\theta^\circ) + O(\delta^3) \qquad (3.2)$$

For the point 7 (Fig.2), $\theta = \theta^\circ = 0$, $\eta_7 = \eta_7^\circ = 0$, $r_7^\circ = \xi_7^\circ$, and according to (3.1) and (3.2),

$$\xi_7^\circ = r_1^\circ + \delta^2\frac{3}{8}\,\frac{(\gamma+1)^2 M_1^8}{m_1^5}\,[(w_{1\alpha})_{\alpha=1,\,\theta=0}]^2 + O(\delta^3) \qquad (3.3)$$

The location of point 7 in the basic coordinate system is calculated from Formula

$$\xi_7 = \frac{\xi_7^\circ - \tan\sigma}{\xi_7^\circ\tan\sigma + 1} \qquad (3.4)$$

According to linear theory

$$(w_{1\alpha})_{\alpha=1,\,\theta=0} = \frac{2}{\pi}\,\frac{\cot\beta}{m_1^2}\csc^2\theta_0 \qquad \left(\cos\theta_0 = \frac{1}{m_1}\cot\beta\right) \qquad (3.5)$$

Using Formulas (3.3) to (3.5) and Tables in [5 and 6], the author calculated ξ_7 for two cases which were also computed numerically using an electronic computer [7]. We cite the results, and also the relative error Δ

γ	M_1	β	δ	$\xi_{7\,[7]}$	$\xi_{7\,(3.4)}$	Δ
1.4	4	30°	5°	0.196	0.199	1.5%
1.4	6	30°	7°	0.116	0.124	7%

4. In this manner, the formation of shocks in exact theory may be predicted from the results of linear theory (i.e. where $(w_\alpha)_{\alpha=1} > 0$, a shock forms, where $(w_\alpha)_{\alpha=1} < 0$, the flow is an expansion). We note that this deduction was first made by Lighthill [8], when he applied his method (PLK) to find the second approximation in conical flow theory. The method presented here is more rigorous; moreover, based on linear theory, we determine the shock position 2-7 to within $O(\delta^2)$, and not to $O(\delta)$ as was done in [8].

The influence of the region where (1.1) is hyperbolic near 2-6-5 and other regions contracting to a point as $\delta \to 0$, are considered in kindred problems in the author's dissertation (Moscow State Univ., 1962).

BIBLIOGRAPHY

1. Bulakh, B.M., K teorii nelineinykh konicheskikh techenii (On the theory of nonlinear conical flows). *PMM* Vol.19, № 4, 1955.

2. Bulakh, B.M., Zamechanie k stat'e L.R.Fauella "Tochnoe i priblizhennoe reshenie dlia sverkhzvukovogo del'taobraznogo kryla" (Remarks on the paper by L.R.Fowell "Exact and approximate solutions for the supersonic delta wing"). *PMM* Vol.22, № 3, 1958.

3. Bulakh, B.M., Zamechanie k stat'e D.V.Reina "Differentsial'no-geometricheskie rassmotreniia preobrazovaniia godografa dlia konicheskogo techeniia" (Remarks on the paper by J.W.Reyn "Differential-geometric considerations of the hodograph transformation for irrotational flow") *PMM* Vol.26, № 4, 1962.

4. Bulakh, B.M., O nekotorykh svoistvakh sverkhzvukovykh konicheskikh techenii gaza (On some properties of supersonic gas flow). *PMM* Vol.25, № 3, 1961.

5. Ferri, A., Aerodynamics of Supersonic Flow. Macmillan, 1949.

6. Piatiznachnye tabiltsy logarifmov chisel i trigonometricheskikh funktsii (Five-place Logarithm Tables of Numbers and Trigonometric Functions). Geodezizdat, 1955.

7. Babaev, D.A., Chislennoe reshenie zadachi obtekaniia verkhnei poverkhnosti treugol'nogo kryla sverkhzvukovym potokom (Numerical solution of the flow past the upper surface of a triangular wing in supersonic flow). Zh.vychisl.Mat.mat.Fiz. Vol.2, № 2, pp.278-289, 1962.

8. Lighthill, M.J., The Shock Strength in Supersonic "Conical Flows". Phil.Mag., Vol.40, ser.7, № 311, p.1202-1223, 1940.

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